EXPLICIT CONSTRUCTION OF MANIFOLDS REALIZING THE PRESCRIBED HOMOLOGY CLASSES

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We consider a well-known N. Steenrod's problem on realization of homology classes by images of the fundamental classes of manifolds. In this paper we give an explicit combinatorial construction that for a given cycle produces a piecewise linear manifold realizing a multiple of the homology class of this cycle. The construction is based on a local procedure for resolving singularities of the cycle. An approach to N. Steenrod's problem based on resolving singularities of pseudo-manifolds was initially proposed by D. Sullivan [1]. D. Sullivan found obstructions to resolving singularities. These obstructions appear to be elements of finite order. Hence D. Sullivan's results imply that one can always resolve singularities "with multiplicity". Our goal is to give an explicit construction for such resolution of singularities.

It is convenient to work with pseudo-manifolds decomposed into *simple cells*. A simple cell is a closed disk whose boundary is decomposed into faces so that the decomposition is dual to a piecewise linear triangulation of a sphere. Rigorous definitions of a simple cell and a decomposition into simple cells can be found in [2]. Examples of decompositions into simple cells are simplicial and cubic decompositions and cell decompositions dual to piecewise linear triangulations of manifolds. All manifolds and pseudo-manifolds are supposed to be closed.

Suppose Z is an oriented n-dimensional pseudo-manifold decomposed into simple cells. The *problem of resolving singularities* of a pseudo-manifold Z is to construct an oriented piecewise linear manifold M and a mapping $g : M \to Z$ such that off the $(n-2)$-dimensional skeleton of Z the mapping g is a finite-fold covering.

Main construction. To each $(n-1)$-dimensional cell of the decomposition Z we assign a label which is an element of a certain finite set. The set of labels of $(n-1)$-dimensional cells containing a cell F is called a *label* of F and is denoted by $c(F)$. A labeling is said to be *good* if

(1) $|c(F)| = n - \dim F$ for any cell F and
(2) for any cell F and any n-dimensional cell $G \supset F$ the labels of all cells H such that $F \subset H \subset G$ are pairwise distinct.

Lemma 1. *Suppose Z is a pseudo-manifold decomposed into simple cells. Then there is a pseudo-manifold $\overline{Z}$ decomposed into simple cells such that $\overline{Z}$ admits a good labeling and there is a mapping $p : \overline{Z} \to Z$ such that p is a covering off the codimension 2 skeleton and p maps each cell of $\overline{Z}$ isomorphically onto a cell of Z.*

Thus in the sequel we may assume that the decomposition Z is endowed with a good labeling. Suppose F is a cell of the decomposition Z such that $\dim F = k < n$. By L_F we denote the set of n-dimensional cells containing F. Since Z admits a good labeling, cells $G \in L_F$ can be regularly colored in two colours. ("Regularly" means that any two cells possessing a common facet are of distinct colours.) Besides, the numbers of cells of both colours are equal to each other. By P_F we denote the set of colour-reversing involutions

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on the set L_F. By P we denote the direct product of the sets P_F over all cells F of the decomposition Z such that $\dim F < n$.

By U we denote the set of $(n+1)$-tuples $(F_0, F_1, \ldots, F_n)$, where $F_0 \supset F_1 \supset \ldots \supset F_n$ are cells of Z and $\dim F_i = n - i$. We put, $V = U \times P \times \mathbb{Z}_2^n$.

We define involutions $\Phi_j^\varepsilon : V \to V$, $\varepsilon = 0, 1$, $j = 1, 2, \ldots, n$, by

$$\Phi_j^0(F_0, F_1, \ldots, F_n, (\Lambda_F), h) = (F_0, F_1, \ldots, F_{j-1}, F_j^*, F_{j+1} \ldots, F_n, (\Lambda_F), h),$$

$$\Phi_j^1(F_0, F_1, \ldots, F_n, (\Lambda_F), h) = (\widetilde{F}_0, \widetilde{F}_1, \ldots, \widetilde{F}_n, (\widetilde{\Lambda}_F), h + e_j),$$

where $\Lambda_F \in P_F$, $h \in \mathbb{Z}_2^n$, $(e_1, e_2, \ldots, e_n)$ is a basis of $\mathbb{Z}_2^n$ and

(1) F_j^* is a unique cell such that $F_j^* \neq F_j$ and $F_{j-1} \subset F_j^* \subset F_{j+1}$;

(2) $\widetilde{F}_i = F_i$ if $i \geqslant j$; $\widetilde{F}_0 = \Lambda_{F_j}(F_0)$; if $0 < i < j$, then $\widetilde{F}_i$ is a unique cell such that $\widetilde{F}_j \subset \widetilde{F}_i \subset \widetilde{F}_0$ and $c(\widetilde{F}_i) = c(F_i)$;

(3) if $c(F) \not\supset c(F_j)$, then $\widetilde{\Lambda}_F = \Lambda_F$;

(4) if $c(F) \supset c(F_j)$ and $G \in L_F$, then $\widetilde{\Lambda}_F(G) = (\Lambda_{H_2} \circ \Lambda_F \circ \Lambda_{H_1})(G)$, where H_1 is a unique cell such that $F \subset H_1 \subset G$ and $c(H_1) = c(F_j)$ and H_2 is a unique cell such that $F \subset H_2 \subset \Lambda_F(\Lambda_{H_1}(G))$ and $c(H_2) = c(F_j)$.

We put, $M = [0,1]^n \times V/\sim$, where $\sim$ is the equivalence relation generated by the identifications

$$(t_1, t_2, \ldots, t_n, \Phi_j^\varepsilon(v)) \sim (t_1, t_2, \ldots, t_n, v) \text{ if } t_j = \varepsilon,\ \varepsilon = 0, 1,\ j = 1, 2, \ldots, n.$$

It can be immediately checked that $\Phi_j^0 \Phi_k^1 = \Phi_k^1 \Phi_j^0$ if $j \neq k$ and $\Phi_j^1 \Phi_k^1 = \Phi_k^1 \Phi_j^1$. This implies that M is an oriented piecewise linear manifold. A required mapping $g : M \to Z$ is given by

$$g(t_1, \ldots, t_n, F_0, \ldots, F_n, (\Lambda_F), h) = \left(\prod_{i=1}^{n} (1 - t_i) \right) b(F_0) + \sum_{j=1}^{n} \left(t_j \prod_{i=j+1}^{n} (1 - t_i) \right) b(F_j),$$

where $b(F)$ is the barycenter of the cell F.

Thus we obtain the following theorem.

Theorem 1. *Main construction yields a mapping $g : M \to Z$ providing a solution to the problem of resolving singularities of a pseudo-manifold Z. Suppose X is a topological space and $f : Z \to X$ is a singular cycle. Then the mapping $f \circ g : M \to X$ realizes a multiple of the homology class of the cycle f.*

As an application of Theorem 1 we give a partial solution of the following problem. Given a set of oriented simplicial spheres $Y_1, Y_2, \ldots, Y_k$ construct an oriented simplicial manifold whose set of links of vertices coincides up to an isomorphism with the set $Y_1, Y_2, \ldots, Y_k$. Such manifold may exist only if vertices of the triangulations $Y_1, Y_2, \ldots, Y_k$ can be paired off so that links of vertices in each pair are isomorphic to each other with the isomorphism reversing the orientation. (If this condition holds, the set $Y_1, Y_2, \ldots, Y_k$ is referred to be *balanced*.)

Theorem 2. *Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of oriented simplicial spheres. There is an explicit construction yielding an oriented simplicial manifold, whose set of links of vertices coincides up to isomorphism with the set*

$$\underbrace{Y_1, \ldots, Y_1}_{r}, \underbrace{Y_2, \ldots, Y_2}_{r}, \ldots, \underbrace{Y_k, \ldots, Y_k}_{r}, K_1, K_2, \ldots, K_l, -K_1, -K_2, \ldots, -K_l,$$

where $-K_i$ is a simplicial sphere K_i with the opposite orientation.

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